Review: Tangent Line - 10/17/16

1 Tangent Line

Definition 1.0.1 *The **tangent line** of f through the point $P = (a, f(a))$ is a line through P with slope*

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

The slope of the tangent line is also called the **instantaneous rate of change**.

Example 1.0.2 *Find the equation for the tangent line of $g(x) = x^2 - 2$ through the point $(1, -1)$. First we find the slope, which is $m = \lim_{x\to 1} \frac{g(x)-g(1)}{x-1} = \lim_{x\to 1} \frac{x^2-2+1}{x-1} = \lim_{x\to 1} \frac{(x+1)(x-1)}{x-1} = \lim_{x\to 1} x + 1 = 2$. Now we can use the slope and our given point to find the equation of the line using point-slope form: $y - (-1) = 2(x - 1) = 2x - 2$, so $y = 2x - 3$.*

If we replace x with $a = h$, we can rewrite our definition of the slope of the tangent line as

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Example 1.0.3 *Use the other definition to find the tangent line of $g(x) = x^2 - 2$ through the point $(1, -1)$. We have $m = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{x\to 0} \frac{1+2h+h^2-2+1}{h} = \lim_{h\to 0} \frac{h(2+h)}{h} = \lim_{h\to 0} 2 + h = 2$. Then we use the point slope form to again get $y = 2x - 3$.*

2 Derivative at a Point

Definition 2.0.4 *The **derivative** of a function f at a point a (denoted $f'(a)$) is the slope of the tangent line to f at the point $(a, f(a))$. That is*

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Note that we can also use our other definition of slope of tangent line, so we can also have

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Example 2.0.5 *Let $f(x) = x^2$. What is $f'(3)$? We have $f'(3) = \lim_{h\to 0} \frac{f(3+h)-f(3)}{h} = \lim_{h\to 0} \frac{9+6h+h^2-9}{h} = \lim_{h\to 0} \frac{h(6+h)}{h} = \lim_{h\to 0} 6 + h = 6$.*

Example 2.0.6 *Let $f(x) = x^2 - 2x$. What is $f'(3)$? Let's try using the other definition. We have $f'(2) = \lim_{x\to 3} \frac{x^2-2x-(9-6)}{x-3} = \lim_{x\to 3} \frac{x^2-2x-3}{x-3} = \lim_{x\to 3} \frac{(x-3)(x+1)}{x-3} = \lim_{x\to 3} x + 1 = 4$.*

Practice Problems

1. Let $f(x) = \frac{7}{x}$. What is $f'(2)$?
2. Let $g(x) = x^2 - 3x$. What is $g'(3)$?
3. Let $k(x) = \frac{3}{x^2}$. What is $k'(1)$?

Solutions

1. We have that $f'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{\frac{7}{2+h} - \frac{7}{2}}{h} = \lim_{h\to 0} \frac{\frac{14}{4+2h} - \frac{14+7h}{4+2h}}{h} = \lim_{h\to 0} \frac{-7h}{h(4+2h)} = \lim_{h\to 0} \frac{-7}{4+2h} = \frac{-7}{4}$.
2. We have that $g'(3) = \lim_{x\to 3} \frac{x^2-3x-(9-9)}{x-3} = \lim_{x\to 3} \frac{x(x-3)}{x-3} = \lim_{x\to 3} x = 3$.
3. We have that $k'(1) = \lim_{x\to 1} \frac{\frac{3}{x^2} - \frac{3}{1}}{x-1} = \lim_{x\to 1} \frac{\frac{3}{x^2} - \frac{3x^2}{x^2}}{x-1} = \lim_{x\to 1} \frac{3-3x^2}{(x-1)(x^2)} = \lim_{x\to 1} \frac{-3(x+1)(x-1)}{(x-1)x^2} = \lim_{x\to 1} \frac{-3(x+1)}{x^2} = \frac{-6}{1} = -6$.